Sequences and Series Cheat Sheet

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Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \dfrac{a}{1-r} & |r| < 1 \\ \\ Diverges & |r| \geq 1 \end{cases}$$

Harmonic Series

$\sum\limits_{i=1}^{\infty} \frac{1}{n}$ This is allways Divergent!

Divergence Test

If $\lim\limits_{x\to\infty} a_x \neq 0$, Then $\sum a_n$ Diverges. Make sure not to misuse this test. This is only for testing Divergence, this does not $\Rightarrow$ convergence

Integral Test

Suppose $f(x)$ is a continuous, positive and decreasing function on the interval, $[K, \infty)$ and that $f(n) = a_n$ then,

1. if $\int\limits_{k}^{\infty} f(x)dx$ is Convergent so is $\sum\limits_{n=k}^{\infty} a_n$
2. if $\int\limits_{k}^{\infty} f(x)dx$ is Divergent so is $\sum\limits_{n=k}^{\infty} a_n$

The lower bound of the integral must be the same as the start of the index for $\sum\limits_{n=k}^{\infty} a_n$

Important to note that the integral test does not give the value of the Series, it simply tells if its Converges/Diverges

Comparison Test

if $0 \leq a_n \leq b_n$ Then:

$\sum_{n=1}^{\infty} a_n$ Diverges $\Rightarrow \sum_{n=1}^{\infty} b_n$ Diverges
$\sum_{n=1}^{\infty} b_n$ Converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ Converges

Limit Comparison

For $\sum a_n$ and $\sum b_n$, where both series contain only positive terms.

If,
$\lim_{n\to\infty} \frac{a_n}{b_n} = C > 0$

Then Either both a_n, b_b Converge or both a_n, b_b Diverge

Alternating Series Test

For $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ with

1) b_n is decreasing
2) b_n is positive
3) $\lim_{n\to\infty} b_x = 0$

Then the series Converges.

Absolute Convergence

$\sum_{n=1}^{\infty} |a_n|$ Converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ Converges.

Absolute Convergence:

$\sum_{n=1}^{\infty} |a_n|$

Conditional Convergence:

$\sum_{n=1}^{\infty} a_n$ Converges.

But

$\sum_{n=1}^{\infty} |a_n|$ Diverges.

Ratio Root Test

1) $\lim_{n\to\infty} |\frac{a_n+1}{a_n}| = L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is Absolutely Convergent.

2) $\lim_{n\to\infty} |\frac{a_n+1}{a_n}| = L > 1$ or Diverges$\Rightarrow \sum_{n=1}^{\infty} a_n$ Diverges.

3) $\lim_{n\to\infty} |\frac{a_n+1}{a_n}| = 1 \Rightarrow$ Inconclusive.

Root Test

1) $\lim_{n\to\infty} \sqrt[n]{a_n} = L < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ is Absolutely Convergent.

2) $\lim_{n\to\infty} \sqrt[n]{a_n} = L > 1$ or Diverges$\Rightarrow \sum_{n=1}^{\infty} a_n$ Diverges.

3) $\lim_{n\to\infty} \sqrt[n]{a_n} = 1 \Rightarrow$ Inconclusive.

Power Series

$\sum_{n=0}^{\infty} c_n(x-a)^n$ Where c_n is a Sequence.

Options for $\sum_{n=0}^{\infty} c_n(x-a)^n$:

1) Diverges unless $x = a$

2) Converges $\forall x$

3) Radius > 0 where:

   i. Converges for $|x-a| < R$

   ii. Diverges for $|x-a| > R$

   iii. More work needed when $|x-a| = R$

MacLaurin Series

If $f(x)$ has a Power Series Representation: $f(x) = \sum_{n=0}^{\infty} c_n \ x^n, \ where \ |x| \ < \ R$

Then $c_n = \frac{f^{(n)}(0)}{n!}$

Taylor Series

If $f(x)$ has a Power Series Representation: $f(x) = \sum_{n=0}^{\infty} c_n \ (x-a)^n$,
$where \ |x - a| \ < \ R$

Then $c_n = \frac{f^{(n)}(a)}{n!}$